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On the mass term of the Dirac equation

Maciej Trzetrzelewski *

NORDITA,
Roslagstullsbacken 23, 106 91 Stockholm,
Sweden

Department of Mathematics,
Royal Institute of Technology,
KTH, 100 44 Stockholm,
Sweden

Abstract

We consider the generalization of the Dirac equation where the mass term is an arbitrary matrix M. A general form of M, consistent with the mass shell constraint, is derived and proven to be equivalent to the original Dirac equation.

1 Introduction

The original way [1] in which Dirac obtained relativistic equation for fermions seems to leave certain ambiguities related to the choice of the mass term. This led some authors [2] to discuss the possibility of generalizing the term by considering certain mass matrices M instead of the usual matrix $m\mathbf{1}$. We would like to point out that consistency conditions actually imply that M must be given by $M = me^{(i\alpha-\beta)\gamma^5}$ with $\alpha \in [0, 2\pi]$ and $\beta \in \mathbb{R}$, of which the cases $\beta = 0$ and $\alpha = 0$ were discussed in [2]. The mass term M can be obtained from the Dirac

*e-mail: maciej.trzetrzelewski@gmail.com

equation by an appropriate change of the phases and the norms of the Weyl spinors.

2 General mass term

Consider a non hermitian x^μ dependent matrix M and assume that the corresponding Dirac equations $D_M\psi = 0$, $D_M = -i\gamma^\mu\partial_\mu + M$ holds. For an arbitrary operator $\mathcal{D}$ the consistency conditions $\mathcal{D}D_M\psi = 0$ have to be satisfied. Due to the mass shell constraint $p_\mu p^\mu = m^2$, $p_\mu = -i\partial_\mu$, useful conditions will come from operators $\mathcal{D}$ which involve the $i\gamma^\mu\partial_\mu$ operator. Let us consider

$$0 = D_{-M}D_M\psi = (m^2 - M^2 - i\gamma^\mu\partial_\mu M)\psi - i[\gamma^\mu, M]\partial_\mu\psi \qquad (1)$$

(we use the conventions $\eta_{\mu\nu} = diag(1,-1,-1,-1)$, $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, $\gamma^5 = +i\gamma^0\gamma^1\gamma^2\gamma^3$). One can also consider other equations e.g.

$$0 = D_{-M^\dagger}D_M\psi = D_M D_M\psi = D_{M^\dagger}D_M\psi = D_{\mathbf{0}}D_M\psi \qquad (2)$$

however as it turns out they do not give new constraints.

If M is equal to $m\mathbf{1}$, equation (1) is trivially satisfied (equations in (2) are either trivial or give the Dirac equation $D_m\psi = 0$). For general M we obtain some nontrivial, first order, differential equations for ψ. These equations must reduce to the Dirac equation $D_M\psi = 0$ - otherwise we would obtain an independent equation for fermions. Concentrating on Eqn. (1) we conclude that

$$[\gamma^\mu, M] = A\gamma^\mu, \qquad (3)$$

$$m^2 - M^2 - i\gamma^\mu\partial_\mu M = AM \qquad (4)$$

for some matrix A.

In order to solve (3) and (4) it is useful to multiply Eqn. (3) from the r.h.s. by γ^μ (no sum) which in particular implies the following equations

$$\gamma^i M\gamma^i - \gamma^j M\gamma^j = 0, \quad 1 \le i < j \le 3.$$

Using explicit representation for gamma matrices we find that the general solution of the letter is

$$M = a(x) + b(x)\gamma^5, \qquad a(x), b(x) \in \mathbb{C} \qquad (5)$$

which using (3) gives $A = -2b(x)\gamma^5$ hence (4) gives

$$m^2 = a(x)^2 - b(x)^2 + \gamma^\mu \partial_\mu \left(a(x) + b(x)\gamma^5 \right). \tag{6}$$

The r.h.s. in (6) should be proportional to the unit matrix hence $\partial_\mu a = \partial_\mu b = 0$. Therefore the general solution of (3) and (4) and hence of the constraint (1) is given by

$$M = a + b\gamma^5, \qquad a, b \in \mathbb{C},$$

$$m^2 = a^2 - b^2. \tag{7}$$

It turns out that (7) also solves other constraint (2). Let us consider the first equation in (2)

$$0 = D_{-M^\dagger} D_M \psi = (m^2 - M^\dagger M - i\gamma^\mu \partial_\mu M)\psi - i(\gamma^\mu M - M^\dagger \gamma^\mu)\partial_\mu \psi. \tag{8}$$

The equations following from (8) are

$$\gamma^\mu M - M^\dagger \gamma^\mu = B\gamma^\mu, \tag{9}$$

$$m^2 - M^\dagger M = BM \tag{10}$$

for some matrix B. Substituting (7) to (9) we find that

$$B = 2i|a| \sin\alpha - 2|b| \cos\beta\gamma^5, \quad \alpha := Arg(a), \quad \beta := Arg(b)$$

which substituted to (10) gives two equations

$$m^2 - |a|^2 - |b|^2 = 2ia|a| \sin\alpha - 2b|b| \cos\beta, \tag{11}$$

$$-2|a||b| \cos(\alpha - \beta) = 2i|a|b \sin\alpha - 2a|b| \cos\beta. \tag{12}$$

Equation (11) is actually equivalent to the second equation in (7) while (12) is an identity hence (8) gives no new constraints on a and b. The same conclusion holds for the remaining equations in (2).

Using the parametrization for the complex circle in (7)

$$a = m(\cos\alpha \cosh\beta - i \sin\alpha \sinh\beta),$$

$$b = mi(\sin\alpha \cosh\beta + i \cos\alpha \sinh\beta)$$

with $\alpha \in [0, 2\pi]$ and $\beta \in \mathbb{R}$ we can write M in the compact form

$$M = me^{(i\alpha - \beta)\gamma^5}. \tag{13}$$

Finally let us observe that this form of M can be obtained from the Dirac equation with $M = m\mathbf{1}$. Noting that in the Weyl representation we have

$$i\sigma^\mu \partial_\mu \psi_L = m e^{-i\alpha+\beta}\psi_R,$$
$$i\bar{\sigma}^\mu \partial_\mu \psi_R = m e^{i\alpha-\beta}\psi_L$$

where ψ_R, ψ_L are Weyl spinors and choosing $\tilde{\psi}_L = e^{\frac{i\alpha-\beta}{2}}\psi_L$, $\tilde{\psi}_R = e^{-\frac{i\alpha-\beta}{2}}\psi_R$ (which could be interpreted as the chiral transformation with the complex angle) the Dirac equation for Weyl spinors $\tilde{\psi}_R$, $\tilde{\psi}_L$ transforms into the standard form with $M = m$.

It is interesting to note that choosing M not belonging to (13) breaks in general the explicit relativistic invariance of equations. As an example let us consider $M = m\gamma_0$. The condition (1) implies that $i\gamma^j\partial_j\psi = 0$ and hence $(i\partial_0 - m)\psi = 0$ which clearly is not Lorentz invariant. This example is particularly interesting (among other choices e.g. $M = \gamma^i$) as there are no negative energy solutions for this choice i.e. the plane-wave ansatz $\psi = ue^{-ikx}$ for positive energy solutions and $\psi = ve^{ikx}$ for negative energy solutions, implies $k_i = 0$, $k_0 = m$ for four basis spinors $[u_s]_t = \delta_{st}$, $s, t = 1, 2, 3, 4$ and no solutions for the v spinor.

3 Conclusions

In this paper we considered generalizations of the Dirac equation where the mass term is replaced by an arbitrary matrix M. It follows that a simple consistency condition (1) implies that M must be of the form (13) which in turn can be obtained from the original Dirac equation by a suitable redefinition of the wavefunction. Therefore the choice $M = m\mathbf{1}$ is already general.

References

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